Modeling, Identification and Control, Vol. 45, No. 3, 2024, pp. 97–103, ISSN 1890–1328

A Stability Study by Routh-Hurwitz Criterion and Gershgorin Circles for Covid-19

B. Akyar [1] A. K. Hansen [2] S. S. Sütlü [3]

[1] *University of Southern Denmark, Department of Mathematics and Computer Science, Campusvej 55, DK-5230, Odense M, Denmark. E-mail:* bediaakyar@gmail.com; bedia@imada.sdu.dk

[2] *Aarhus University, Department of Mathematics, Ny Munkegade 118, DK-8000, Aarhus C, Denmark E-mail:* aysekara66@gmail.com; akhansen@au.dk

[3] *Haliç University, Faculty of Arts and Science, Department of Mathematics, 34060 Eyüpsultan, Istanbul, Turkiye. E-mail:* selcuksultan@gmail.com; sultansutlu@halic.edu.tr

Abstract

In this paper, we study stabilization problem on a model for covid-19 by using Routh-Hurwitz criterion and Gershgorin circles. Using Routh-Hurwitz criterion, we prove the necessity of unstability and stability conditions for the model that we extend from an existing one. We give the necessary conditions for stability on this model by using the Gershgorin Circle Theorem and give examples.

Keywords: Routh-Hurwitz criterion, stability, unstability, Gershgorin circle.

1 Introduction

In this work, we study stability problem on a model for covid-19 by using Routh-Hurwitz criterion and Gershgorin circles. The model that we study in this paper can also be considered for any other infectious diseases.

Chronic or autoimmune diseases such as cancer and diabetes are described as the diseases of our age. However, infectious diseases still remain a problem for all humanity. Recently, epidemic diseases and their effects have come to the fore again with the coronavirus outbreak. The mathematical theory of infectious diseases and epidemics has always been interesting for many branches of science. Research on this subject has accelerated as it has become even more interesting with the covid-19 epidemic, which has affected almost the whole world in recent years.

It has been observed that it is very important to be able to predict the future behavior of the disease in terms of understanding the spread, termination, or general effects of infectious diseases on the population they infect. In this regard, it is clear that mathematical modeling of epidemics as dynamical systems is very useful for controlling epidemics, interpreting and reducing their consequences.

The first mathematical model and formalization in epidemiology was written by Daniel Bernoulli for smallpox in 1760, Bernoulli (1766). In Dietz and Heesterbeek (2002), Bernoulli's work is revisited in modern mathematical language and dynamics divided into two disjoint compartments: the suspectibles and immunes. Besides, it can be said that a deterministic model literally started in the 20th century. In the article Kermack and Mckendrick (1927), a general theory for infectious diseases was given using ordinary differential equation systems, and mathematical epidemiology has shown an exponential development since then. Enormous varity of mathematical models have been put forward for many infectious diseases, and their analyzes and various applications have been made, Hethcote (2000).

Mathematical models are generally expressed with

doi:10.4173/mic.2024.3.2

abbreviated names such as SI, SIR, SIS, SEI, SIS, SIRS, SEIS, SIRS, SEIR, SEIRS by dividing the population into various groups. S indicates the susceptible class, I is the class of infectives, E is the class of exposed, R is the class of recovered individuals. There are also models that include classes representing individuals hospitalized with H, individuals vaccinated with V, intubated class with N and classes M representing infants with passive immunity.

In our work, we use XEIHR model which we extend and modify from a model given in Koker and Gozukizil (2021), where we denote by X the population class, E the class of exposed, I the class of infectives, H the hospitalized class and R the class of recovered individuals.

This paper consists of six sections. In the second section, we explain our model which is extended and modified by the model given in Koker and Gozukizil (2021). In the third section, we study stabilization problem on our extended model for covid-19 by using Routh-Hurwitz criterion, prove the necessity of unstability and stability conditions for this model and we finish the section with two examples. In the fourth section, we give the necessary conditions for stability on our extended model by using the Gershgorin Circle Theorem, and two examples of unstability and stability on our model, respectively. We give the discussion in the fifth section and lastly the conclusion of the paper.

2 An extended model for Covid-19

In this section, we want to explain the dynamical system that we use for studying the stabilization problem. We consider the model of covid-19 given in Koker and Gozukizil (2021) by the following family of differential equations with some modifications on the parameter γ by taking as all real numbers and on one class, and we denote the system of these differential equations by (1):

$$\begin{aligned}\frac{dX}{dt} &= \lambda^* - \mu X - \beta XE \\ \frac{dE}{dt} &= \beta XE - \epsilon EI - \mu E \\ \frac{dI}{dt} &= \epsilon EI - (\mu+\omega+\gamma)I \\ \frac{dH}{dt} &= \omega I - (\mu+\alpha+\delta)H \\ \frac{dR}{dt} &= \gamma I + \delta H - \mu R\end{aligned} \tag{1}$$

Here, $X(t)$ denotes the population, $E(t)$ is the exposed class, $I(t)$ is the infected (infectious) class, $H(t)$ is the hospitalized class and $R(t)$ is the recovered class. About parameters, λ^* is the birth rate per capita, μ is the natural death rate per capita, α is the average death rate from the covid-19 virus, β is the rate which moves from the population to the exposed class, ϵ is the rate of progression from exposure to infection, ω is the rate at which the infected become hospitalized, γ is the rate of recovery of those infected and δ is the recovery rate of hospitalized patients. In Koker and Gozukizil (2021), the authors consider $H(t)$ as the intubated (severely ill) class which is denoted by $N(t)$ and all parameters are positive.

We have the diagram in Figure 1. which corresponds to equations (1)

In this work, we consider a hospitalized class H (not necessary intubated) and $\gamma \in \mathbb{R}$. In this model, γ has three possibilities, where positive case goes directly to the recovered class, negative case goes to the hospitalized class and if it is zero it remains in the same class. The disease-free equilibrium point is $E_0 = (\frac{\lambda^*}{\mu}, 0, 0, 0, 0)$

In Koker and Gozukizil (2021), the authors give the reproduction number $\frac{\beta\lambda^*}{\mu^2}$ which is the dominant eigenvalue of the matrix FV^{-1}, where F and V are the following matrices:

$$F = \begin{pmatrix} \beta X & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2}$$

and

$$V = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu+\omega+\gamma & 0 \\ 0 & -\omega & \mu+\alpha+\gamma \end{pmatrix} \tag{3}$$

Here, we calculate details due to the fact that we will use some outcomes through the calculation.

Let $x = (E, I, H) = (x_1, x_2, x_3)$. Then, we have the following two matrices from the differential equations given above:

$$\mathcal{F} = \begin{pmatrix} \beta XE \\ 0 \\ 0 \end{pmatrix} \tag{4}$$

and

$$\mathcal{V} = \begin{pmatrix} \epsilon EI + \mu E \\ -\epsilon EI + (\mu+\omega+\gamma)I \\ -\omega I + (\mu+\alpha+\gamma)H \end{pmatrix} \tag{5}$$

such that

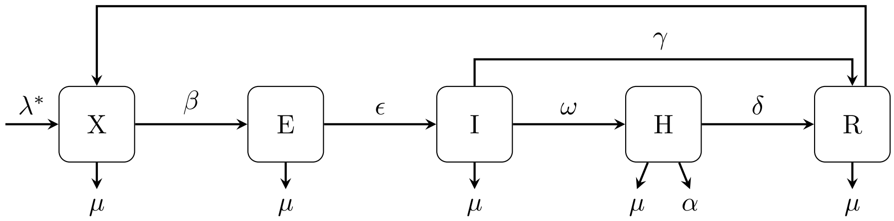

Figure 1: Schematic Diagram of the Model

$$\frac{dx}{dt} = \mathcal{F}(x_1,x_2,x_3) - \mathcal{V}(x_1,x_2,x_3) \quad (6)$$

$$= \begin{pmatrix} \beta X x_1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \epsilon x_1 x_2 + \mu x_1 \\ -\epsilon x_1 x_2 + (\mu+\omega+\gamma)x_2 \\ -\omega x_2 + (\mu+\alpha+\gamma)x_3 \end{pmatrix} \quad (7)$$

Hence, F is the Jacobian matrix of $\mathcal{F}$ and V is the Jacobian matrix of $\mathcal{V}$ at the disease-free equilibrium point E_0. Moreover,

$$V^{-1} = \begin{pmatrix} \frac{1}{\mu} & 0 & 0 \\ 0 & \frac{1}{\mu+\omega+\gamma} & 0 \\ 0 & \frac{\omega}{(\mu+\omega+\gamma)(\mu+\alpha+\gamma)} & \frac{1}{\mu+\alpha+\gamma} \end{pmatrix} \quad (8)$$

which implies $\mu + \alpha + \gamma = 1$.

The reproduction number is independent of the hospitalized class H and in the following section, we will consider this information for the differential equations.

3 Routh-Hurwitz Criterion Approach for Stabilization

In this section, we study the stabilization of the model that we explain above by using Routh-Hurwitz criterion. We know that the reproduction number is independent of w. Therefore, we reduce the dynamical system which corresponds to (1) to the disease area out of the hospital class H for stabilization and consider the system of the following equations which we denote by (9)

$$\begin{aligned} \frac{dX}{dt} &= \lambda^* - \mu X - \beta X \\ \frac{dE}{dt} &= \beta XE - \epsilon EI - \mu EE \\ \frac{dI}{dt} &= \epsilon EI - (\mu+\gamma)I \end{aligned} \quad (9)$$

Then, the Jacobian matrix $Jac(X,E,I)$ is equal to

$$\begin{pmatrix} -\mu-\beta E & -\beta X & 0 \\ \beta E & \beta X - \epsilon I - \mu & -\epsilon E \\ 0 & \epsilon I & \epsilon E - (\mu+\gamma) \end{pmatrix} \quad (10)$$

The disease-free equilibrium point is $E_0^* = (\frac{\lambda^*}{\mu}, 0, 0)$, and the Jacobian matrix at this point is

$$Jac_{(E_0^*)}(X,E,I) = \begin{pmatrix} -\mu & \frac{-\beta\lambda^*}{\mu} & 0 \\ 0 & \frac{\beta\lambda^*}{\mu} - \mu & 0 \\ 0 & 0 & -(\mu+\gamma) \end{pmatrix}. \quad (11)$$

Then, the characteristic polynomial will be

$$|\lambda I_3 - Jac_{(E_0^*)}(X,E,I)| = \lambda^3 + (3\mu - \frac{\beta\lambda^*}{\mu} + \gamma)\lambda^2 + (3\mu^2 - 2\beta\lambda^* - \frac{\beta\lambda^*\gamma}{\mu} + 2\mu\gamma)\lambda - \beta\lambda^*(\mu+\gamma) + \mu^3 + \mu^2\gamma, \quad (12)$$

where I_3 is the 3×3 identity matrix.

3.1. Corollary:

Let us consider the system of equations (9) . If $\beta\lambda^* > \mu^2$, then the dynamical system defined by these equations is unstable at the disease-free equilibrium point $E_0^* = (\frac{\lambda^*}{\mu}, 0, 0)$, so is the dynamical system defined by (1) .

Proof: According to the necessary conditions for Routh-Hurwitz criterion all coefficients have to have the same sign and for our case they all have to be at positive sign. Therefore, $-\beta\lambda^*(\mu+\gamma) + \mu^3 + \mu^2\gamma < 0$ implies unstability of the system.

3.2. Theorem:

If $\beta\lambda^* < \mu^2$, then the dynamical system defined by (10) is stable at the disease-free equilibrium point $E_0^* = (\frac{\lambda^*}{\mu}, 0, 0)$.

Proof: We have the following table by using the characteristic polynomial:

$$\begin{array}{lcc} \lambda^3: & 1 & 3\mu^2 - 2\beta\lambda^* - \frac{\beta\lambda^*\gamma}{\mu} + 2\mu\gamma \\ \lambda^2: & 3\mu - \frac{\beta\lambda^*}{\mu} + \gamma & -\beta\lambda^*(\mu+\gamma) + \mu^3 + \mu^2\gamma \\ \lambda^1: & a_1 & a_2 \\ \lambda^0: & b_1 & b_2 \end{array}$$

and with simple calculation we get

$$\begin{aligned} a_1 &= \frac{(3\mu^2 - 2\beta\lambda^* - \frac{\beta\lambda^*\gamma}{\mu} + 2\mu\gamma)(3\mu - \frac{\beta\lambda^*}{\mu} + \gamma)}{3\mu - \beta\frac{\lambda^*}{\mu} + \gamma} \\ &+ \frac{\beta\lambda^*(\mu+\gamma) - \mu^3 - \mu^2\gamma}{3\mu - \beta\frac{\lambda^*}{\mu} + \gamma} \\ b_1 &= \frac{a_1(-\beta\lambda^*(\mu+\gamma) + \mu^3 + \mu^2\gamma)}{a_1} \\ a_2 &= 0 \\ b_2 &= 0 \end{aligned}$$

By the Routh-Hurwitz criterion, elements in the first column of the table should have the same sign. To show that $3\mu - \frac{\beta\lambda^*}{\mu} + \gamma > 0$ for stability, we use the first condition and have the following:

$$3\mu^2 - \beta\lambda^* + \gamma\mu > 2\mu^2 + \gamma\mu > 0. \tag{13}$$

In fact, $\mu > 0$ and $\mu + \gamma > 0$. In order to have a_1 as a positive number, it is enough to guarantee that

$$3\mu^2 - 2\beta\lambda^* - \frac{\beta\lambda^*\gamma}{\mu} + 2\mu\gamma > 0 \tag{14}$$

By using the first condition, we can write

$$3\mu^2 - 2\beta\lambda^* - \frac{\beta\lambda^*\gamma}{\mu} + 2\mu\gamma > \mu^2 + \mu\gamma. \tag{15}$$

$b_1 > 0$ and the proof is complete.

3.3. Example:

Let us consider a dynamical system governed by the following differential equations:

$$\begin{aligned} \frac{dX}{dt} &= \frac{1}{10} - 2X - XE \\ \frac{dE}{dt} &= XE - EI - 2E \\ \frac{dI}{dt} &= EI - I \end{aligned} \tag{16}$$

Then, the Jacobian matrix at the disease-free equilibrium point is

$$Jac_{(E_0^*)}(X,E,I) = \begin{pmatrix} -2 & -\frac{1}{20} & 0 \\ 0 & -\frac{39}{20} & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{17}$$

and the characteristic polynomial is

$$|\lambda I_3 - Jac_{(E_0^*)}(X,E,I)| = \lambda^3 + \frac{99}{20}\lambda^2 + \frac{157}{20}\lambda + \frac{39}{10} \tag{18}$$

By the Routh-Hurwitz criterion, we have the following table:

$$\begin{array}{lcc} \lambda^3: & 1 & \frac{157}{20} \\ \lambda^2: & \frac{99}{20} & \frac{39}{10} \\ \lambda^1: & a_1 & a_2 \\ \lambda^0: & b_1 & b_2 \end{array} \tag{19}$$

and with simple calculation we get $a_1 = 7.06\bar{2}1,\ a_2 = 0,\ b_1 = 3.9\ \textit{and}\ b_2 = 0$.

Since all the elements in the first column of the table are positive, this system is stable at the disease-free equilibrium point (E_0^*).

In this example, we can see that the stability of the system is directly related with the square of the natural death rate per capita (μ) as being strictly greater than the multiplication of the birth rate per capita (λ^*) and the rate which moves from the population to the exposed class (β).

3.4. Example:

In this example, we use the Turkey's data of 03.07.2021 given in the reference Koker and Gozukizil (2021) and consider the dynamical system governed by the following differential equations:

$$\begin{aligned} \frac{dX}{dt} &= 0.013 - 0.053X - 0.078XE \\ \frac{dE}{dt} &= 0.078XE - 0.023EI - 0.053E \\ \frac{dI}{dt} &= 0.023EI - 0.94I \end{aligned} \tag{20}$$

Then, the Jacobian matrix at the disease-free equilibrium point $E_0^* = (0.245, 0, 0)$ is

$$Jac_{(E_0^*)}(X,E,I) = \begin{pmatrix} -0.053 & -0.019 & 0 \\ 0 & -0.034 & 0 \\ 0 & 0 & -0.94 \end{pmatrix} \tag{21}$$

and the characteristic polynomial is

$$|\lambda I_3 - Jac_{(E_0^*)}(X,E,I)| = \lambda^3 + 1.027\lambda^2 + 0.084\lambda + 0.002 \tag{22}$$

By the Routh-Hurwitz criterion, we have the following table:

λ^3 :	1	0.084
λ^2 :	1.027	0.002
λ^1 :	a_1	a_2
λ^0 :	b_1	b_2

and with simple calculation we get $a_1 = 0.082, a_2 = 0, b_1 = 0.002$ and $b_2 = 0$.

Since all the elements in the first column of the table are positive, this system is stable at the disease-free equilibrium point (E_0^*).

4 Examples of Gershgorin Circles for Unstability and Stability

In this section, we study examples for stability and unstability for the dynamical system determined by the (9) differential equations. We know that the Jacobian matrix at the disease-free equilibrium point is

$$Jac_{(E_0^*)}(X,E,I) = \begin{pmatrix} -\mu & \frac{-\beta\lambda^*}{\mu} & 0 \\ 0 & \frac{\beta\lambda^*}{\mu} - \mu & 0 \\ 0 & 0 & -(\mu+\gamma) \end{pmatrix} \tag{23}$$

By the Corollary of Gershgorin Circle Theorem, Adom-Konadu et al. (2022), we have the following result.

4.1 Corollary:

If $\mu > -\gamma$ and $\mu > \frac{\beta\lambda^*}{\mu}$, then the eigenvalues of $Jac_{(E_0^*)}(X,E,I)$ matrix are negative or have negative real parts.

Moreover, it is proved in Bejarano et al. (2018) that, if the Jacobian matrix of the dynamical system determined by the (9) at the disease-free equilibrium point is strictly diagonally dominant, then by the Gershgorin Circle Theorem the disease-free equilibrium point is locally asymptotically stable.

4.2. Example:

Let us consider a dynamical system governed by the following differential equations:

$$\begin{aligned} \frac{dX}{dt} &= \frac{1}{9} - 2X - \frac{1}{2}XE \\ \frac{dE}{dt} &= \frac{1}{2}XE - EI - 2EE \\ \frac{dI}{dt} &= EI + I \end{aligned} \tag{24}$$

Then, the Jacobian matrix at the disease-free equilibrium point is

$$Jac_{(E_0^*)}(X,E,I) = \begin{pmatrix} -2 & -\frac{1}{36} & 0 \\ 0 & -\frac{71}{36} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{25}$$

Here, one of the eigenvalues of $Jac_{(E_0^*)}(X,E,I)$ matrix is positive. In fact, $\mu = 2 < -\gamma = -(-3)$. Then, the system is unstable at the disease-free equilibrium point E_0^*.

In this example, we can see that the unstability of the system is directly related with the sum of the natural death rate per capita (μ) and the rate of recovery of those infected (γ) which is stricly negative, that is, the absolute value of the rate of recovery of those infected has to be strictly greater than the natural death rate per capita.

Gershgorin circle is drawn in the following

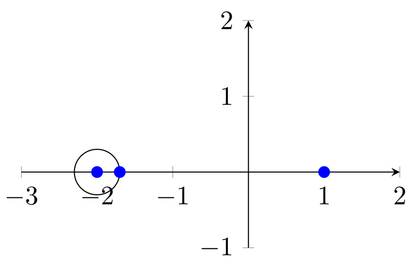

Figure 2: Gershgorin circles of unstable system in 4.2 Example

We can see that one of the eigenvalues lies on the right of the x-axis. Here, we use approximated numbers so that circle can be seen since some numbers are too small to illustrate.

4.3. Example:

Let us consider a dynamical system governed by the following differential equations:

$$\begin{aligned} \frac{dX}{dt} &= \frac{1}{9} - \frac{1}{3}X - \frac{1}{2}XE \\ \frac{dE}{dt} &= \frac{1}{2}XE - EI - \frac{1}{3}E \\ \frac{dI}{dt} &= EI - \frac{8}{15}I \end{aligned} \tag{26}$$

Then, the Jacobian matrix at the disease-free equilibrium point is

$$Jac_{(E_0^*)}(X,E,I) = \begin{pmatrix} -\frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & -\frac{8}{15} \end{pmatrix} \tag{27}$$

Here, $\mu = \frac{1}{3} > -\gamma = -\frac{1}{5}$ and $\mu = \frac{1}{3} > \frac{\beta\lambda^*}{\mu} = \frac{1}{6}$ (i.e.; the reproduction number $\frac{\beta\lambda^*}{\mu^2} = \frac{1}{2} < 1$) and all the eigenvalues of $Jac_{(E_0^*)}(X,E,I)$ matrix are negative. It follows that the disease-free equilibrium point E_0^* is locally asymptotically stable, Bejarano et al. (2018).

In the contrary of the previous example, we can see that the locally asymptotically stability of the system is directly related with the sum of the natural death rate per capita (μ) and the rate of recovery of those infected (γ) which is positive, that is, the natural death rate per capita has to be strictly greater than the absolute value of the rate of recovery of those infected.

Two of the Gershgorin circles are degenerate circles in this example, too, and all of them are drawn as follows;

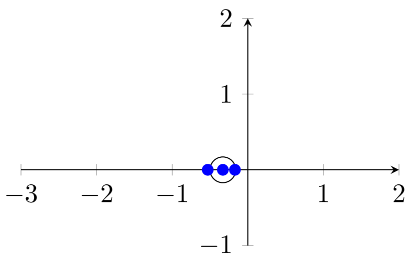

Figure 3: Gershgorin circles of stable system in 4.3 Example

and we can see that all of the eigenvalues lie on the left of the x-axis.

Eigenvalues play a crucial role for Gershgorin circles and therefore for unstability and local asymptotically stability decision of the dynamical systems. In this section we illustrated these two cases, i.e., unstability and local asymptotically stability decision by two examples.

5 Discussion

One of the biggest concerns with any infectious disease is its ability to spread through a population and epidemic diseases and their effects have come to the fore again with the coronavirus outbreak. Mathematical modeling of epidemics as dynamical systems is very useful for controlling epidemics, interpreting and reducing their consequences. Therefore, we study stability problem on our XEIHR model which we extend and modify from a model given in Koker and Gozukizil (2021), where we denote by X the population class, E the class of exposed, I the class of infectives, H the hospitalized class and R the class of recovered individuals. We use Routh- Hurwitz criterion to prove the necessity of unstability and stability conditions for our model since it is a classical approach for this kind of systems and easy to predict on numerical examples. We use the Turkey's data of 03.07.2021 given in the reference Koker and Gozukizil (2021) as well as randomly chosen numerical data in examples. Although this provides the opportunity for comparison with real-world data, the data here is limited for generalizations in our world. Nevertheless, we characterize mathematically so that anyone who has interest can calculate with her/his data and our results are good contribution to the area. Moreover, we give the necessary conditions for stability on our extended model by using the Gershgorin Circle Theorem, and we draw geometrically the Gershgorin circles which are degenerate circles in two examples one unstability and another stability in our model, respectively. This kind of models can also be considered for any other infectious diseases.

6 Conclusion

In this work, we studied stability problem using XEIHR model which we extended and modified from the model given in Koker and Gozukizil (2021), using Routh- Hurwitz criterion and Gershgorin circles. Here, X is the population class, E is the class of exposed, I is the class of infectives, H is the hospitalized class and R is the class of recovered individuals. We considered the first three equations for the Routh- Hurwitz criterion, since the reproduction number is independent of the hospitalized class H, and calculated the characteristic polynomial of the Jacobian matrix. Then, we characterized unstability and stability for this dynamical system (9) . In order to prove these characterizations, we used Routh- Hurwitz criterion. We finished this section by giving two examples that one with randomly chosen numerical data and the other one with the Turkey's data of 03.07.2021 given in the reference Koker and Gozukizil (2021). In the fourth section, we gave the necessary conditions for stability on the dynamical system (9) by using the Gershgorin Circle Theorem, and finished the study with two examples with randomly chosen numerical data for unstability and stability on our model, respectively. Finally, we discussed our results and examples.

References

Adom-Konadu, A., Sackitey, A. L., and Anokye, M. Local stability analysis of epidemic models using a corollary of gershgorin's circle theorem. *Applied Mathematics E-Notes, Research Square*, 2022. doi:https://doi.org/10.21203/rs.3.rs-1909006/v1.

Bejarano, D., Ibargüen-Mondragón, E., and Gómez-Hernández, E. A stability test for non linear systems of ordinary differential equations based on the gershgorin circles. *Contemporary Engineering Science*, 2018. 11(91):4541–4548. doi:10.12988/ces.2018.89504.

Bernoulli, D. Essai d'une nouvelle analyse de la mortalite causee par la petite verole. *Mem. Math. Phys. Acad. Roy.Sci., Paris*, 1766. pages 1, (Reprinted in: L.P. Bouckaert, B.L. van der Waerden (Eds.), Die Werke von Daniel Bernoulli, Bd. 2 Analysis und Wahrscheinlichkeitsrechnung, Birkhauser, Basel, 1982, p. 235. English translation entitled 'An attempt at a new analysis of the mortality caused by smallpox and of the advantages of inoculation to prevent it' in: L. Bradley, Smallpox Inoculation: An Eighteenth Century Mathematical Controversy, Adult Education Department, Nottingham, 1971, p. 21. Reprinted in: S. Haberman, T.A. Sibbett (Eds.) History of Actuarial Science, vol. VIII, Multiple Decrement and Multiple State Models, William Pickering, London, 1995, p. 1.).

Dietz, K. and Heesterbeek, J. Daniel bernoulli's epidemiological model revisited. *Mathematical Biosciences*, 2002. 180(1:-2):1–21. doi:10.12988/ces.2018.89504.

Hethcote, H. W. The mathematics of infectious diseases. *Society for Industrial and Applied Mathematics*, 2000. 42(4):599–653. doi:10.1137/S0036144500371907.

Kermack, W. and Mckendrick, A. The mathematics of infectious diseases. *A Contribution to the Mathematical Theory of Epidemics,Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Characters*, 1927. 115(772):700–721. doi:10.1098/rspa.1927.01187.

Koker, N. and Gozukizil, O. Application of an epidemic model to turkey data and stability analysis for the covid-19 pandemic. *Sakarya University Journal of Science SAUJS*, 2021. 25(6):1438–1445. doi:10.16984/saufenbilder.980797.